

# HIGHER SYMMETRIES OF THE LAPLACIAN VIA QUANTIZATION

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**ABSTRACT.** We develop a new approach, based on quantization methods, to study higher symmetries of invariant differential operators. We focus here on conformally invariant powers of the Laplacian over a conformally flat manifold, and recover results of Eastwood, Leistner, Gover and Silhan.

In particular, conformally equivariant quantization establishes a crystal clear correspondence between hamiltonian symmetries of the null geodesic flow and the algebra of higher symmetries of the conformal Laplacian. Resorting to symplectic reduction, this leads to a quantization of the minimal nilpotent coadjoint orbit of the conformal group and allows to identify the latter algebra of symmetries in terms of the Joseph ideal. By the way, we obtain a tangential star-product for a family of coadjoint orbits of the conformal group.

## 1. INTRODUCTION

The higher symmetries of a differential operator  $P$  are the differential operators  $D_1$  satisfying  $PD_1 = D_2P$  for some differential operator  $D_2$ . They clearly form an algebra and preserve the kernel of  $P$ . The first order ones define a Lie algebra  $\mathfrak{g}$  generalizing the invariance Lie algebra of  $P$ . The determination of the space of higher symmetries of  $P$ , together with its algebra and  $\mathfrak{g}$ -module structure, is of interest from at least two point of view: the integrability of the equation  $P\phi = 0$ , with  $\phi$  in the source space of  $P$ , and the representation theory of the  $\mathfrak{g}$ -module  $\ker P$ .

The first example which deserves to be investigated is certainly the conformal (or Yamabe) Laplacian, that we consider on a conformally flat manifold  $(M, g)$  of arbitrary signature  $(p, q)$  and dimension  $n = p + q$ . It writes as  $\Delta = \nabla_i g^{ij} \nabla_j + \frac{n-2}{4(n-1)}R$ , where  $\nabla$  is the Levi-Civita connection and  $R$  the scalar curvature, and it sends  $\lambda$ - to  $\mu$ -densities, with  $\lambda = \frac{n-2}{2n}$ ,  $\mu = \frac{n+2}{2n}$ . On one hand,  $\Delta\phi = 0$  is the most basic wave equation, describing e.g. a free massless quantum particle on  $(M, g)$ . Its integration has been achieved in various contexts resorting to symmetries, and we highlight here only two seminal works. On  $\mathbb{R}^3$ , Boyer, Kalnins and Miller have classified all the second order symmetries of the Laplacian [8], which allows them to get all the possible coordinates systems separating the equation  $\Delta\phi = 0$ . For curved but Ricci-flat manifolds, Carter has built up a way to get symmetries, by quantization of Killing tensors [11], and has successfully used it to integrate on Kerr space-time the wave equation  $(\Delta + m^2)\phi = 0$ , with a massive term  $m \in \mathbb{R}$ . On the other hand, the first order

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symmetries of the conformal Laplacian are given by

$$\Delta(X + \lambda \text{Div} X) = (X + \mu \text{Div} X) \Delta,$$

where  $\text{Div}$  is the divergence operator and  $X \in \mathfrak{g} \simeq \mathfrak{o}(p+1, q+1)$  is a conformal Killing vector. This provides a representation of  $\mathfrak{g}$  on  $\ker \Delta$  which integrates, if  $M = \mathbb{S}^p \times \mathbb{S}^q$  and  $p+q$  even, to a unitary irreducible representation of the Lie group  $G = \text{O}(p+1, q+1)$  [5]. This is the intensively studied minimal representation of  $G$ , see e.g. [26, 27]. It is linked to the minimal nilpotent coadjoint orbit  $\mathcal{O}_0$  of  $G$  through the Joseph ideal [24], which appears as the kernel of the representation of  $\mathfrak{U}(\mathfrak{g})$ , the universal enveloping algebra of  $\mathfrak{g}$ , on  $\ker \Delta$ . Nevertheless, this representation cannot be obtained via the Kirillov's orbit method [25] since  $\mathcal{O}_0$  admits no invariant polarization [40]. In the complex setting, the alternative deformation program has been applied successfully by Arnal, Benamor and Cahen, which have proved existence and uniqueness of a graded  $\mathfrak{g}$ -equivariant star product on regular functions on  $\mathcal{O}_0^{\mathbb{C}}$  [1]. From this star product, Astashkevich and Brylinski have built a unitary irreducible representation of  $G^{\mathbb{C}}$  [2], but the way to recover the real representation of  $G$  on  $\ker \Delta$  is unclear.

Resorting to conformal ambient space, Eastwood has been able to determine the space of higher symmetries of the conformal Laplacian [18]. The latter corresponds as  $\mathfrak{g}$ -module to the space of conformal Killing tensors, which are the Hamiltonian symmetries of the null geodesic flow. Besides, as an algebra, it identifies to a quotient  $\mathfrak{U}(\mathfrak{g})/J$  of the universal enveloping algebra. Surprisingly, this ideal  $J$  coincides with the previously mentioned Joseph ideal [17]. The correspondence obtained by Eastwood might then be interpreted as a quantization of the minimal coadjoint orbit of  $G$ .

Our aim is precisely to prove that the results of Eastwood stem from the now well-developed theory of equivariant quantization of cotangent bundles [15, 7, 32, 10], and that after a basic symplectic reduction, this quantization provides a supplement to the orbit method in the case of the minimal nilpotent coadjoint orbit of  $G$ . In particular, it induces on that orbit the star product of Arnal, Benamor and Cahen. The present approach can be generalized to number of cases, indeed, equivariant quantization is available for any  $|1|$ -graded parabolic geometry and for differential operators acting on any irreducible natural bundles [10]. To illustrate its efficiency, we deal with the determination of higher symmetries of the conformal powers of the Laplacian, denoted  $\Delta^\ell$ . Thus, we recover recent results of Gover and Silhan [23] obtained via tractor calculus and highly non-trivial computations. These are up to our knowledge the only cases which have been fully worked out. Let us now detail the content of this article.

In Section 2, we introduce our main tools, namely the classification of conformally invariant operators on symbols [20, 4, 33], the conformally equivariant quantization [15] and the induced star product on symbols [14].

In Section 3 lies our first main result. We characterize the space  $\mathcal{A}^{\lambda, \ell}$  of higher symmetries of  $\Delta^\ell$  and the space  $\mathcal{K}^\ell$  of  $\ell$ -generalized conformal Killing tensors [34] as kernels of conformally invariant operators. Then, we prove that conformally equivariant quantization maps one

operator to the other, and then one space to the other. This correspondence can be made explicit, since formulae have been derived for this quantization [16, 31], even in the curved case [36, 38].

In section 4, we first introduce algebras  $\mathcal{K}$  and  $\mathcal{A}^\lambda$  generated by  $\mathfrak{g}$ , the Lie algebra of conformal vector fields. The spaces of symmetries  $\mathcal{K}^\ell$  and  $\mathcal{A}^{\lambda,\ell}$  are obtained as quotients of the latter. Then, we describe all the coadjoint orbits of  $G$  in the image of the moment map  $\mu : T^*\mathbb{R}^{p+1,q+1} \rightarrow \mathfrak{g}^*$  as symplectic reductions of the source manifold. Their algebras of regular functions are determined, and two of them identify to  $\mathcal{K}$  and  $\mathcal{K}^1$ . As a consequence, we get an explicit description of the symmetry algebras  $\mathcal{A}^{\lambda,\ell}$ . In particular, the algebra  $\mathcal{K}^1$  is associated to the minimal nilpotent coadjoint orbit  $\mathcal{O}_{00}$  and the corresponding symmetry algebra  $\mathcal{A}^{\lambda,1}$  of the Laplacian is isomorphic to the universal enveloping algebra of  $\mathfrak{g}$  moded out by the Joseph ideal. Finally, we built a star product on each coadjoint orbit in the image of  $\mu$  and discuss the quantization of the nilpotent ones. In particular, the conformally equivariant quantization induces the star-product of Arnal, Benamor and Cahen [1] on  $\mathcal{O}_{00}$  and furnishes a representation of it on  $\ker \Delta$ , which integrates into the minimal representation of  $G$  if  $M = \mathbb{S}^p \times \mathbb{S}^q$  and  $p + q$  is even.

## 2. CONFORMAL GEOMETRY OF DIFFERENTIAL OPERATORS AND OF THEIR SYMBOLS

We introduce in this section the basic notions that we need and the two key facts leading to our main theorem in the next section, namely: the existence and uniqueness of the conformally equivariant quantization [15] and the classification of the conformally invariant operators on the space of symbols [33].

**2.1. Actions of vector fields.** Let  $M$  be a smooth manifold. We start with the definitions of the algebra  $\mathcal{D}(M)$  of differential operators on  $M$  and of its algebra of symbols  $\mathcal{S}(M)$ . The first one is filtered by the subspaces  $\mathcal{D}_k(M)$  of differential operators of order  $k$ , defined as the spaces of operators  $A$  on  $\mathcal{C}^\infty(M)$  satisfying  $[\cdots[A, f_0], \cdots], f_k] = 0$  for all functions  $f_0, \dots, f_k \in \mathcal{C}^\infty(M)$ . The second one,  $\mathcal{S}(M)$ , is its canonically associated graded algebra, defined by  $\mathcal{S}(M) = \bigoplus_{k=0}^\infty \mathcal{D}_k(M)/\mathcal{D}_{k-1}(M)$ . It identifies to the algebra of functions on  $T^*M$  which are polynomial in the fibers, the grading corresponding to the polynomial degree. The canonical projection  $\sigma_k : \mathcal{D}_k(M) \rightarrow \mathcal{D}_k(M)/\mathcal{D}_{k-1}(M)$  is called the principal symbol map.

We are interested in the action of the Lie algebra  $\text{Vect}(M)$  of vector fields on these both algebras. The diffeomorphisms of  $M$  lift canonically into automorphisms of  $\text{GL}(M)$ , the principal bundle of linear frames over  $M$ . Consequently, they act canonically on sections of every associated bundles to  $\text{GL}(M)$ , and their infinitesimal actions induce the one of the Lie algebra  $\text{Vect}(M)$ . Hence, we get a  $\text{Vect}(M)$ -module structure on the space of sections  $\mathcal{F}^\lambda$  of the trivial line bundle  $|\Lambda^n T^*M|^{\otimes \lambda}$ , with  $\lambda \in \mathbb{R}$ . This is a one parameter deformation of the module  $\mathcal{C}^\infty(M)$ . Indeed, via a global section  $|\text{vol}|$ , it identifies to the module  $(\mathcal{C}^\infty(M), \ell^\lambda)$ , with the  $\text{Vect}(M)$ -action  $\ell_X^\lambda = X + \lambda \text{Div}(X)$ , the operator  $\text{Div}$  being the divergence w.r.t.  $|\text{vol}|$ . The elements of  $\mathcal{F}^\lambda$  are named tensor densities of weight  $\lambda$ , or  $\lambda$ -densities for short. Via the

adjoint action, it gives rise to the  $\text{Vect}(M)$ -module  $\mathcal{D}^{\lambda,\mu}$  of differential operators from  $\lambda$ - to  $\mu$ -densities, which identifies to  $(\mathcal{D}(M), \mathcal{L}^{\lambda,\mu})$ , with  $\mathcal{L}_X^{\lambda,\mu} A = \ell_X^\mu A - A \ell_X^\lambda$ , for all  $X \in \text{Vect}(M)$  and  $A \in \mathcal{D}(M)$ . This action preserves the filtration of  $\mathcal{D}(M)$ , hence the algebra of symbols inherits of a  $\text{Vect}(M)$ -action compatible with the grading. This action coincides with the canonical one on the functions on  $T^*M$  tensorized with  $\delta$ -densities, for  $\delta = \mu - \lambda$ . We denote by  $\mathcal{S}^\delta$  the obtained module of symbols and by  $\mathcal{S}_k^\delta$  the submodule of homogeneous symbols of degree  $k$ .

**2.2. Conformal Lie algebra.** A conformal structure on a smooth manifold  $M$  is given by the equivalence class  $[g]$  of a pseudo-Riemannian metric  $g$ , where two metrics  $h$  and  $g$  are considered equivalent if  $h = Fg$  for some positive function  $F \in \mathcal{C}^\infty(M)$ . The signature  $(p, q)$  of the metric  $g$  is an invariant of the conformal structure, and to each signature corresponds a canonical flat model  $(\mathbb{R}^{p,q}, [\eta])$ , with  $\eta = \mathbb{I}_p \otimes -\mathbb{I}_q$ . If there exists an atlas  $(U_i, \phi_i)$  on  $(M, [g])$  such that the pull-back by every chart  $\phi_i$  of the canonical flat conformal structure coincides with the restriction of  $[g]$  to  $U_i$ , the conformal manifold  $(M, [g])$  is said to be conformally flat. The vector fields that preserve a conformal class  $[g]$  are called conformal Killing vector fields, they satisfy  $L_X g = Fg$ , with  $L_X g$  the Lie derivative of  $g$  along  $X$  and  $F$  a non-negative function. They form a sheaf of Lie algebras, and if  $(M, [g])$  is conformally flat, it is locally isomorphic to  $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$  the conformal Lie algebra of  $(\mathbb{R}^{p,q}, [\eta])$ .

An important example of conformally flat manifold is  $\mathbb{S}^p \times \mathbb{S}^q$ , viewed as a homogeneous space of  $G = \text{O}(p+1, q+1)$ . Starting from the isometric action of  $G$  on the pseudo-Euclidean space  $\mathbb{R}^{p+1, q+1}$ , we get an action of  $G$  on the space of isotropic half-lines, which identifies naturally to the manifold  $\mathbb{S}^p \times \mathbb{S}^q$ . Via this construction, the flat metric on  $\mathbb{R}^{p+1, q+1}$  induces a conformally flat structure on  $\mathbb{S}^p \times \mathbb{S}^q$ , preserved by the  $G$ -action.

**2.3. Conformal invariants.** Over a conformally flat manifold  $(M, [g])$  of signature  $(p, q)$ , a conformal invariant is a differential geometric object which is invariant under the action of conformal Killing vector fields. First, we provide the classification of the conformal invariants of the  $\text{Vect}(M)$ -modules of differential operators  $\mathcal{D}^{\lambda,\mu}$  and of symbols  $\mathcal{S}^\delta$ , see e.g. [35]. Using local conformal coordinates  $(x^i)$ , which are such that  $g_{ij} = F\eta_{ij}$  for a positive function  $F$ , this amounts to perform the classification over the flat space. We can then provide explicit local expressions in terms of the local conformal coordinates  $(x^i, p_i)$  on  $T^*M$ , and their corresponding derivatives  $(\partial_i, \partial_{p_i})$ .

**Proposition 2.3.1.** *On a conformally flat manifold  $(M, [g])$ , the conformal invariants of  $(\mathcal{S}^\delta)_{\delta \in \mathbb{R}}$  and  $(\mathcal{D}^{\lambda,\mu})_{\lambda, \mu \in \mathbb{R}}$  are given locally, up to a multiplicative constant, by*

- $R^\ell$  for  $k \in \mathbb{N}$  and  $\delta = \frac{2\ell}{n}$ ,
- $\Delta^\ell$  for  $k \in \mathbb{N}$  and  $\lambda = \frac{n-2\ell}{2n}$ ,  $\mu = \frac{n+2\ell}{2n}$ ,

where  $R = \eta^{ij} p_i p_j$ , and  $\Delta = \eta^{ij} \partial_i \partial_j$  is the conformal Laplacian.

We refer to [22] and references inside for global expressions of the conformal powers of the Laplacian. Since the principal symbol map is  $\text{Vect}(\mathbb{R}^n)$ -equivariant, conformally invariant

differential operators give rise to conformally invariant symbols, but the fact that they are in correspondence is remarkable. Second, we present the classification of the conformally invariant differential operators on the space of symbols, which arises from [20, 4] and is explicitly described in [33]. This relies on the harmonic decomposition of the  $\mathfrak{g}$ -module of symbols, namely  $\mathcal{S}^\delta = \bigoplus_{k,s \in \mathbb{N}, 2s \leq k} \mathcal{S}_{k,s}^\delta$ , where  $\mathcal{S}_{k,s}^\delta$  is the submodule of homogeneous symbols of degree  $k$  of the form  $P = R^s Q$  with  $Q$  a traceless symbol, i.e.  $TQ = 0$  for  $T = \eta_{ij} \partial_{p_i} \partial_{p_j}$ .

**Theorem 2.3.2.** [33] *Let  $k \geq 2s$  and  $k' \geq 2s'$  be integers, and  $\delta, \delta' \in \mathbb{R}$ . The space of conformal invariant differential operators:  $\mathcal{S}_{k,s}^\delta \rightarrow \mathcal{S}_{k',s'}^{\delta'}$ , is either trivial or of dimension 1. In the latter case  $j = \frac{n}{2}(\delta' - \delta)$  is an integer and the space is generated locally by*

- $R^{s'} D^d T^s$ , if  $s' - s = j$ ,  $k - k' = d - 2j$  and  $\delta = 1 + \frac{2(k-s)-d-1}{n}$ ,
- $R^{s'} G_0^g T^s$ , if  $g + s' - s = j$ ,  $k - k' = s - s' - j$  and  $\delta = \frac{2s+1-g}{n}$ ,
- $R^{s'} \mathcal{L}_\ell T^s$ , if  $\ell + s' - s = j$ ,  $k - k' = 2(\ell - j)$  and  $\delta = \frac{1}{2} + \frac{k-\ell}{n}$ ,

where  $D = \partial_i \partial_{p_i}$  is the divergence operator,  $G_0$  is the gradient operator  $\eta^{ij} p_i \partial_j$  composed with the projection on traceless symbols and  $\mathcal{L}_\ell$  is an avatar of  $\ell^{\text{th}}$  power of Laplacian.

Global expression for divergence and gradient operators can be found in [13], and we refer to [42] for  $\mathcal{L}_1$ .

**2.4. Conformally equivariant quantization.** Let  $\lambda, \mu \in \mathbb{R}$  and  $\delta = \mu - \lambda$ . We call quantization the linear isomorphisms  $\mathcal{Q} : \mathcal{S}^\delta \rightarrow \mathcal{D}^{\lambda,\mu}$  which are right inverse of the principal symbol map on homogeneous symbols. If the both  $\text{Vect}(M)$ -modules of symbols and of differential operators are indeed isomorphic as vector spaces by the very definition of symbols, they are not as  $\text{Vect}(M)$ -modules [28, 29]. A natural question is whether they are isomorphic as modules over given Lie subalgebras of  $\text{Vect}(M)$ , and whether this extra structure allows to single out one quantization. This has been studied in various contexts under the name *equivariant quantization*, and one of the fundamental result is the following.

**Theorem 2.4.1.** [15] *On a conformally flat manifold, there exists a unique conformally equivariant quantization for generic values of  $\delta = \mu - \lambda$ , i.e. a unique  $\mathfrak{g}$ -module morphism  $\mathcal{Q}^{\lambda,\mu} : \mathcal{S}^\delta \rightarrow \mathcal{D}^{\lambda,\mu}$  which is the right inverse of the principal symbol map on homogeneous symbols.*

The exceptional values of  $\delta$  leading to a non-unique or a non-existing conformally equivariant quantization have been classified in [38, 33]. They can be obtained via Theorem 2.3.2 and the following result.

**Proposition 2.4.2.** [33] *The conformally equivariant quantization exists and is unique on  $\mathcal{S}_{k,s}^\delta$  if and only if there is no conformally invariant differential operators from  $\mathcal{S}_{k,s}^\delta$  to  $\mathcal{S}^\delta$ .*

In particular  $\delta = 0$  is not an exceptional value. Several works have been devoted to the obtention of explicit formulae for the conformally equivariant quantization. Restricting to the space  $\mathcal{S}_{*,0}^\delta = \bigoplus_{k \in \mathbb{N}} \mathcal{S}_{k,0}^\delta$  of traceless symbols, Radoux [36] has derived an explicit formula in

the curved case, i.e. on any manifold endowed with a conformal class of metrics. We give it here in the flat case as an example, resorting to the divergence operator  $D = \partial_i \partial_{p_i}$  and the normal ordering  $\mathcal{N} : P^{i_1 \dots i_k}(x) p_{i_1} \dots p_{i_k} \mapsto P^{i_1 \dots i_k}(x) \partial_{i_1} \dots \partial_{i_k}$ .

**Proposition 2.4.3.** [36] *Let  $\delta \notin \{1 + \frac{2k-1-m}{n} \mid m = 1, \dots, k\}$ . On the space  $\mathcal{S}_{k,0}^\delta$  of traceless symbols of degree  $k$ , the conformally equivariant quantization is given by*

$$(2.1) \quad \mathcal{Q}^{\lambda,\mu} = \mathcal{N} \circ \left( \sum_{m=0}^k c_m^k D^m \right),$$

with  $c_0^k = 1$  and  $c_m^k = \frac{k-m+n\lambda}{m(2k-m-1+n(1-\delta))} c_{m-1}^k$ , for  $m = 1, \dots, k$ .

In the general case, including symbols with non-vanishing trace, fully explicit formulae are known only for symbols up to the order 3 in momenta variables  $p$ , they are available in conformal coordinates [30] as well as in covariant terms [16, 31]. Finally, let us mention that Silhan has obtained the expression of the conformally equivariant quantization in the curved case on all symbols, resorting to tractor calculus [38].

**2.5. Conformally equivariant graded star product.** Let us start with standard definitions. The algebra of symbols  $\mathcal{S}^0$  is commutative and graded, moreover, as a subalgebra of  $\mathcal{C}^\infty(T^*M)$ , it carries a Poisson bracket denoted by  $\{\cdot, \cdot\}$ . A graded (or homogeneous) star product on  $\mathcal{S}^0$  is an associative  $\mathbb{C}[[\hbar]]$ -linear product  $\star$  on  $\mathcal{S}^0 \otimes \mathbb{C}[[\hbar]]$ , with  $\hbar$  a formal parameter. For  $P, Q \in \mathcal{S}^0$ , it is of the form  $P \star Q = \sum_{m \in \mathbb{N}} (i\hbar)^m B_m(P, Q)$  and satisfies:

- (1)  $B_0(P, Q) = PQ$ ,
- (2)  $B_1(P, Q) - B_1(Q, P) = \{P, Q\}$ ,
- (3) for all integers  $k, l, m$ ,  $B_m : \mathcal{S}_k^0 \otimes \mathcal{S}_l^0 \rightarrow \mathcal{S}_{k+l-m}^0$  is a bidifferential operator.

The latter hypothesis does not imply that  $B_m$  is a bidifferential operator on  $\mathcal{S}^0 \otimes \mathcal{S}^0$ , and this will indeed not be the case in the following, see [2, 14]. A frequently required extra property is the symmetry (or parity) of the star product, namely  $B_m(P, Q) = (-1)^m B_m(Q, P)$  for all integers  $m$ , or equivalently  $\overline{P \star Q} = \overline{Q} \star \overline{P}$ , where  $\bar{\cdot}$  is the complex conjugation.

One source of star products on  $\mathcal{S}^0$  is quantizations. Let us introduce two maps, the linear map  $\mathfrak{S} : \mathcal{S}^0 \rightarrow \mathcal{S}^0 \otimes \mathbb{C}[[\hbar]]$  defined by  $(i\hbar)^k \text{Id}$  on  $\mathcal{S}_k^0$  and the  $\mathbb{C}[[\hbar]]$ -linear extension of some quantization  $\mathcal{Q} \otimes \text{Id} : \mathcal{S}^0 \otimes \mathbb{C}[[\hbar]] \rightarrow \mathcal{D}^{\lambda,\lambda} \otimes \mathbb{C}[[\hbar]]$ . Clearly, the composition  $\mathcal{Q}_\hbar = (\mathcal{Q} \otimes \text{Id}) \circ \mathfrak{S}$  gives rise to a graded star product on  $\mathcal{S}^0$  as the pull back by  $\mathcal{Q}_\hbar$  of the composition of differential operators, i.e.  $P \star Q = \mathcal{Q}_\hbar^{-1}(\mathcal{Q}_\hbar(P) \circ \mathcal{Q}_\hbar(Q))$ . Moreover, for  $\lambda = \frac{1}{2}$ , this star product is symmetric iff the quantization satisfies  $\mathcal{Q}_\hbar(\overline{P}) = \mathcal{Q}_\hbar(P)^*$  for all  $P \in \mathcal{S}^0$ . Here, the superscript  $*$  denotes the adjoint operation w.r.t. the Hermitian product on complex compactly supported half-densities, given by  $(\phi, \psi) = \int_M \overline{\phi} \psi$ .

The previously defined Hamiltonian action of  $\text{Vect}(M)$  on the algebra of symbols  $\mathcal{S}^0$  leads to a comoment map  $X = X^i \partial_i \mapsto \mu_X = X^i p_i$  and the action of  $X \in \text{Vect}(M)$  on  $P \in \mathcal{S}^0$  reads then as  $\{\mu_X, P\}$ . Thanks to the star product on  $\mathcal{S}^0$ , we can define a new action of  $\text{Vect}(M)$  on  $\mathcal{S}^0$  via the star bracket, i.e.  $X \in \text{Vect}(M)$  acts on  $P \in \mathcal{S}^0$  by  $[\mu_X, P]_\star = \mu_X \star P - P \star \mu_X$ .

The star product is said conformally equivariant (or strongly  $\mathfrak{g}$ -invariant) if both induced  $\mathfrak{g}$ -actions coincide, namely  $[\mu_X, P]_\star = i\hbar\{\mu_X, P\}$  for all  $X \in \mathfrak{g}$ .

The star product  $\star_\lambda$  induced by the conformally equivariant quantization  $\mathcal{Q}^{\lambda,\lambda}$  (see Theorem 2.4.1) is obviously a graded  $\mathfrak{g}$ -equivariant star product on  $\mathcal{S}^0$ . Conversely, conditions under which a  $\mathfrak{g}$ -equivariant symmetric star product on  $\mathcal{S}^0$  comes from conformally equivariant quantization are derived explicitly in [14]. For completeness, let us state a similar but weaker result without parity requirement. It will not be use in the following.

**Proposition 2.5.1.** *Let  $(M, [\mathfrak{g}])$  be a conformally flat manifold and  $\star$  be a graded  $\mathfrak{g}$ -equivariant star product on  $\mathcal{S}^0$ . There exists  $\lambda \in \mathbb{R}$  such that  $\star = \star_\lambda$ , and it is symmetric iff  $\lambda = \frac{1}{2}$ .*

*Proof.* The strategy is to show that  $\star$  comes from some quantization  $\mathcal{Q}$ , and then to identify this quantization with the conformally equivariant quantization  $\mathcal{Q}^{\lambda,\lambda}$  valued in  $\mathcal{D}^{\lambda,\lambda}$ .

For  $f$  a function and  $X$  a vector field, we get  $f \star \mu_X = f\mu_X + i\hbar B_1(f, X)$ . Working locally, in conformal coordinates, the  $\mathfrak{g}$ -equivariance of  $B_1$  on  $\mathcal{S}_0^0 \otimes \mathcal{S}_1^0$  implies that it is equal to the bidifferential operator  $\lambda \partial_i \otimes \partial_{p_i}$  for some  $\lambda \in \mathbb{R}$ . Consequently, we have  $f \star \mu_X = f\mu_X + i\hbar \lambda X(f)$ .

Now, we define the linear map  $\Psi : \mathcal{D}_1(M) \rightarrow \mathcal{S}^0 \otimes \mathbb{C}[[\hbar]]$  by the identity on  $\mathcal{C}^\infty(M)$  and by  $X \mapsto \mu_X - i\hbar \lambda \text{Div}(X)$  on  $\text{Vect}(M)$ . This is a Lie algebra morphism for respectively the commutator and the star bracket. Since  $\text{Div}(fX) = f\text{Div}(X) + X(f)$ , the map  $\Psi$  is also  $\mathcal{C}^\infty(M)$ -linear, the action being given by multiplication w.r.t. the composition and the star product respectively. Then,  $\Psi$  can be extended as an algebra morphism between  $\mathcal{D}(M) \otimes \mathbb{C}[[\hbar]]$  and  $(\mathcal{S}^0 \otimes \mathbb{C}[[\hbar]], \star)$ . The latter admits an inverse  $\mathcal{Q}_h$ , which is clearly a quantization whose induced star product is  $\star$ . Its  $\mathfrak{g}$ -equivariance ensures  $\mathcal{Q}_h(\{\mu_X, P\}) = \frac{1}{i\hbar}[\mathcal{Q}_h(\mu_X), \mathcal{Q}_h(P)]$  for all  $P \in \mathcal{S}^0$  and  $X \in \mathfrak{g}$ . Since  $\mathcal{Q}_h(\mu_X) = i\hbar \ell_X^\lambda$ , the underlying map  $\mathcal{Q}$  is a  $\mathfrak{g}$ -equivariant quantization valued in  $\mathcal{D}^{\lambda,\lambda}$ . By uniqueness statement in Theorem 2.4.1,  $\mathcal{Q}$  is equal to  $\mathcal{Q}^{\lambda,\lambda}$ , and then  $\star = \star_\lambda$ . For parity statement see [15].  $\square$

### 3. CLASSIFICATION OF THE HIGHER SYMMETRIES OF THE CONFORMAL POWERS OF THE LAPLACIAN

The aim of this section is to show how conformally equivariant quantization sheds new light on the determination of higher symmetries of conformal Laplacian, initiated by Eastwood [18] and pursued in [19] and [23] for conformal powers of the Laplacian, in the conformally flat case. In all this section we work over a conformally flat manifold  $(M, [\mathfrak{g}])$  and  $\Delta^\ell$  denotes the conformal  $\ell^{\text{th}}$  power of the Laplacian, pertaining to  $\mathcal{D}^{\lambda,\mu}$  for values of the weights henceforth fixed to  $\lambda = \frac{n-2\ell}{2n}$ ,  $\mu = \frac{n+2\ell}{2n}$ .

**3.1. Definition of higher symmetries of  $\Delta^\ell$ .** There is a number of different notions of symmetries for a differential operator, let us first discuss some of them for  $\Delta^\ell = (\eta^{ij} \partial_i \partial_j)^\ell$  on the flat space  $(\mathbb{R}^n, \eta)$ . The most basic one is given by vector fields  $X \in \text{Vect}(\mathbb{R}^n)$  preserving the considered operator:  $[\Delta^\ell, X] = 0$ . We can broaden this notion by considering

rather differential operators  $D \in \mathcal{D}(\mathbb{R}^n)$  commuting to  $\Delta^\ell$ . They are called *higher symmetries*, in contradistinction with vector fields which are only first order differential operators. Such symmetries preserve obviously the eigenspaces of  $\Delta^\ell$ . Here we are interested in more general higher symmetries preserving only its kernel. Hence, those inside the left ideal  $(\Delta^\ell) = \{P\Delta^\ell \mid P \in \mathcal{D}(\mathbb{R}^n)\}$  are considered trivial. Resorting to conformal coordinates, higher symmetries prove to be locally the same on flat and conformally flat manifolds. Global existence can nevertheless be problematic in this more general setting, hence all the following statements should be understood either locally or in terms of sheaves.

**Definition 3.1.1.** *Let  $\lambda = \frac{n-2\ell}{2n}$ ,  $\mu = \frac{n+2\ell}{2n}$  and let  $\Delta^\ell \in \mathcal{D}^{\lambda,\mu}$  be the conformal  $\ell^{\text{th}}$  power of the Laplacian on conformally flat manifold  $(M, [g])$ . A higher symmetry of  $\Delta^\ell$  is a class of differential operators  $[D_1] \in \mathcal{D}^{\lambda,\lambda}/(\Delta^\ell)$ , such that  $\Delta^\ell D_1 = D_2 \Delta^\ell$ , for some  $D_2 \in \mathcal{D}^{\mu,\mu}$ .*

This definition does not depend on the chosen representative since  $\Delta^\ell(P\Delta^\ell) = (\Delta^\ell P)\Delta^\ell$  for any differential operator  $P$ . We denote by  $\mathcal{A}^{\lambda,\ell}$  the space of higher symmetries of  $\Delta^\ell$ . It turns to be an algebra that is characterized as the kernel of the conformally invariant map

$$(3.1) \quad \begin{aligned} \text{QHS} : \mathcal{D}^{\lambda,\lambda}/(\Delta^\ell) &\rightarrow \mathcal{D}^{\lambda,\mu}/(\Delta^\ell) \\ [D] &\mapsto [\Delta^\ell D] \end{aligned}$$

Let us mention that the left ideal generated by  $\Delta^\ell$  can be defined in the modules  $\mathcal{D}^{\lambda,\lambda'}$  for any  $\lambda'$ , and in particular for  $\lambda' = \mu$ , as the subspace  $(\Delta^\ell) = \{P\Delta^\ell \mid P \in \mathcal{D}^{\mu,\lambda'}\}$ .

**Example 3.1.2.** *The higher symmetries of  $\Delta^\ell$  given by first order differential operators are the Lie derivatives  $\ell_X^\lambda$  for  $X \in \mathfrak{g}$ , in accordance with Proposition 2.3.1 we have  $\Delta^\ell \ell_X^\lambda = \ell_X^\mu \Delta^\ell$ .*

**3.2. Symmetries of the null geodesic flow and generalizations.** Using the properties of the principal symbol map  $\sigma$ , the equality  $[D, \Delta] = A\Delta$  defining higher symmetries of the Laplacian implies  $\{\sigma(D), R\} = \sigma(A)R$ , where  $\{\cdot, \cdot\}$  denotes the canonical Poisson bracket on  $T^*M$ . Consequently, the principal symbol of higher symmetries of the Laplacian is a constant of motion for the null geodesic flow. The latter are known to be given by conformal Killing tensors, which are symmetric tensors and identify to symbols of weight 0. In the following, round bracket denotes symmetrization of indices,  $\nabla$  the Levi-Civita connection and  $L$  is an arbitrary tensor.

**Definition 3.2.1.** *A conformal Killing  $k$ -tensor  $K$  is defined equivalently as*

- *A symmetric traceless tensor of order  $k$  s.t.  $\nabla_{(i_0} K_{i_1 \dots i_k)} = g_{(i_0 i_1} L_{i_2 \dots i_k)}$ ,*
- *A traceless symbol of degree  $k$  satisfying  $\{R, K\} = fR$  for some function  $f$ ,*
- *A traceless symbol of degree  $k$  in the kernel of  $G_0$ .*

For  $k = 1$ , we recover the notion of conformal Killing vectors. The conformal Killing tensors of higher orders correspond to transformations of the phase space  $T^*M$  not preserving the configuration manifold  $M$ . Resorting to Theorem 2.3.2, the operator  $G_0 : \mathcal{S}_{k,0}^0 \rightarrow \mathcal{S}_{k+1,0}^{\frac{2}{n}}$  is conformally invariant for every  $k$ , and this is the only one on  $\mathcal{S}_{k,0}^0$ . Hence, the space of



conformal Killing tensors of a given order is an indecomposable representation space of  $\mathfrak{g}$ . Furthermore, this space happens to be finite dimensional, see e.g. [13] and references inside. The semi-simplicity of  $\mathfrak{g}$  allows then to conclude that the space of conformal Killing tensors of a given order is an irreducible representation of  $\mathfrak{g}$ . We can generalize this picture to tensors (or symbols) with trace, using the conformal invariance of  $G_0^{2s+1}T^s$  on  $\mathcal{S}_{k,s}^0$ . The following definition is due to Nikitin and Prilipko [34].

**Definition 3.2.2.** *A  $s$ -generalized conformal Killing  $k$ -tensor  $K$  is defined equivalently as*

- *A symmetric traceless tensor of order  $(k-2s)$  s.t.  $\nabla_{(i_0} \cdots \nabla_{i_{2s}} K_{i_{2s+1} \cdots i_k)} = g_{(i_0 i_1} L_{i_2 \cdots i_k)}$ ,*
- *A symbol in  $\mathcal{S}_{k,s}^0$  which is in the kernel of  $G_0^{2s+1}T^s$ .*

Note that we use the multiplication by the symbol  $R^s$  to pass from the first to the second assertion in the definition above, and we apply the operator  $T^s$  for the opposite way. Again, the space of  $s$ -generalized conformal Killing tensors of order  $k$  is an irreducible representation of  $\mathfrak{g}$  [23]. We denote this subspace of  $\mathcal{S}_{k,s}^0$  by  $\mathcal{K}_{k,s}$  and we introduce the spaces of classical higher symmetries  $\mathcal{K}^\ell = \bigoplus_{s=0}^{\ell-1} \mathcal{K}_{*,s}$ , with  $\mathcal{K}_{*,s} = \bigoplus_{k \geq 2s} \mathcal{K}_{k,s}$ , as well as their limit  $\mathcal{K} = \bigoplus_{s \in \mathbb{N}} \mathcal{K}_{*,s}$ .

**3.3. From classical to quantum symmetries.** Gover and Silhan have proved that the two types of symmetries defined in the two previous paragraphs are in bijection [23]. We propose a completely different and far less technical proof, showing that the latter correspondence is given by equivariant quantization. We assume that  $\ell \in \mathbb{N}^*$ ,  $\lambda = \frac{n-2\ell}{2n}$  and  $\mu = \frac{n+2\ell}{2n}$ .

**Theorem 3.3.1.** *The conformally equivariant quantization descends to an isomorphism of  $\mathfrak{g}$ -modules  $\mathcal{Q}^{\lambda,\lambda} : \mathcal{K}^\ell \rightarrow \mathcal{A}^{\lambda,\ell}$ , identifying higher symmetries of  $\Delta^\ell$  with  $s$ -generalized conformal Killing tensor for  $s < \ell$ . Moreover, every  $P \in \mathcal{K}$  satisfies  $\Delta^\ell \mathcal{Q}^{\lambda,\lambda}(P) = \mathcal{Q}^{\mu,\mu}(P) \Delta^\ell$ .*

*Proof.* The uniqueness of the conformally equivariant quantization leads to the following factorization  $\mathcal{Q}^{\lambda,\lambda'}(PR^\ell) = \mathcal{Q}^{\mu,\lambda'}(P)\Delta^\ell$ , for any symbol  $P$  and any  $\lambda' \in \mathbb{R}$ . Consequently, we get an isomorphism of  $\mathfrak{g}$ -modules on the quotient spaces  $\mathcal{Q}^{\lambda,\lambda'} : \bigoplus_{s=0}^{\ell-1} \mathcal{S}_{*,s}^{\lambda'-\lambda} \rightarrow \mathcal{D}^{\lambda,\lambda'} / (\Delta^\ell)$ , where  $\mathcal{S}_{*,s}^\delta = \bigoplus_{k \in \mathbb{N}} \mathcal{S}_{k,s}^\delta$ . We keep notation  $\mathcal{Q}^{\lambda,\lambda'}$  in the latter case for simplicity.

The idea of the proof is to use the conformally equivariant quantization to identify the kernel  $\mathcal{A}^{\lambda,\ell}$  of the operator QHS, see (3.1), to the one of an operator CHS on symbols. The latter is determined such that the following diagram of  $\mathfrak{g}$ -modules is commutative

$$\begin{array}{ccc} \mathcal{D}^{\lambda,\lambda} / (\Delta^\ell) & \xrightarrow{\text{QHS}} & \mathcal{D}^{\lambda,\mu} / (\Delta^\ell) \\ \mathcal{Q}^{\lambda,\lambda} \uparrow & & \uparrow \mathcal{Q}^{\lambda,\mu} \\ \bigoplus_{s=0}^{\ell-1} \mathcal{S}_{*,s}^0 & \xrightarrow{\text{CHS}} & \bigoplus_{s=0}^{\ell-1} \mathcal{S}_{*,s}^{\frac{2\ell}{n}} \end{array}$$

The quantization  $\mathcal{Q}^{\lambda,\lambda}$  always exists but not  $\mathcal{Q}^{\lambda,\mu}$ . Let us suppose first that  $n$  is odd or  $3\ell \leq \frac{n}{2} + 1$ . Then  $\mathcal{Q}^{\lambda,\mu}$  exists and the operator CHS is well-defined and conformally invariant. Resorting to the classification given in Theorem 2.3.2, it is necessarily equal on  $\mathcal{S}_{*,s}^0$  to  $R^{\ell-s-1}G_0^{2s+1}T^s$ , up to a multiplicative constant. This constant can not be zero since QHS

does not vanish on the image of  $\mathcal{S}_{*,s}^0$ . Hence, by Definition 3.2.2, the kernel of QHS is isomorphic to the space  $\mathcal{K}^\ell$  of  $s$ -generalized conformal Killing tensors for  $0 \leq s < \ell$ . Now, we can define on  $\mathcal{D}^{\lambda,\lambda}$  a new conformally invariant operator  $\text{QHS}_0 : D \mapsto \Delta^\ell D - \mathcal{Q}^{\mu,\mu} \circ (\mathcal{Q}^{\lambda,\lambda})^{-1}(D) \Delta^\ell$ , and via the commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\lambda,\lambda} & \xrightarrow{\text{QHS}_0} & \mathcal{D}^{\lambda,\mu} \\ \mathcal{Q}^{\lambda,\lambda} \uparrow & & \uparrow \mathcal{Q}^{\lambda,\mu} \\ \mathcal{S}^0 & \xrightarrow{\quad} & \mathcal{S}^{\frac{2\ell}{n}} \end{array}$$

we get a non-vanishing conformally invariant operator on  $\mathcal{S}_{*,s}^0$  for every  $s$ . According to Theorem 2.3.2, it is proportional to CHS if  $s < \ell - 1$  and this is the null operator else. We conclude that, in particular,  $\Delta^k \mathcal{Q}^{\lambda,\lambda}(P) = \mathcal{Q}^{\mu,\mu}(P) \Delta^k$  for any  $P \in \mathcal{K}$ .

Let us now treat the general case. We resort to the principal symbol map to define a conformally invariant operator on symbols via the following commutative diagram

$$(3.2) \quad \begin{array}{ccc} \mathcal{D}_k^{\lambda,\lambda}/(\Delta^\ell) & \xrightarrow{\text{QHS}} & \mathcal{D}_{k'}^{\lambda,\mu}/(\Delta^\ell) \\ \mathcal{Q}^{\lambda,\lambda} \uparrow & & \downarrow \sigma_{k'} \\ \mathcal{S}_{k,s}^0 & \xrightarrow{\text{CHS}_\sigma} & \mathcal{S}_{k'}^{\frac{2\ell}{n}}, \end{array}$$

where  $k' \in \mathbb{N}$  is taken as small as possible, so that  $\text{CHS}_\sigma$  does not vanish. The principal symbol map is not an isomorphism, hence, a priori, we just get that the space  $\mathcal{Q}^{\lambda,\lambda}(\ker \text{CHS}_\sigma)$  contains the kernel of QHS. If  $\ell$  is big enough and  $k$  small enough to avoid negative exponents, Theorem 2.3.2 leads now to two possibilities:  $\text{CHS}_\sigma$  is proportional to  $R^{\ell-s-1} G_0^{2s+1} T^s$  or to  $R^{\ell-s-k-\frac{n}{2}} \mathcal{L}_{k+\frac{n}{2}} T^s$ . We restrict then diagram (3.2) to these kernels and get a principal symbol  $\sigma_{k''}$ , with  $k''$  a strictly smaller integer than  $k'$ , but the possible invariant operators are still the same. Then, the kernel of QHS corresponds to either the one of  $R^{\ell-s-1} G_0^{2s+1} T^s$  or the one of  $R^{\ell-s-k-\frac{n}{2}} \mathcal{L}_{k+\frac{n}{2}} T^s$ . The intersection does not occur as the kernel of the first operator is irreducible. Besides, the kernel of the second operator is infinite dimensional. But the kernel of QHS is an algebra, and for  $k$  high enough  $\sigma_k(\mathcal{A}^{\lambda,\ell})$  is the kernel of  $R^{\ell-s-1} G_0^{2s+1} T^s$ , which is finite dimensional. In conclusion,  $\text{CHS}_\sigma$  is proportional to  $R^{\ell-s-1} G_0^{2s+1} T^s$  on any  $\mathcal{S}_{k,s}^0$  and we get the desired correspondence in the general case.  $\square$

**Remark 3.3.2.** For  $\lambda = \frac{n-2}{2n}$ , classical and quantum symmetries for the equations of motion of a free massless particle correspond to each other via  $\mathcal{Q}^{\lambda,\lambda} : \{R, K\} \propto R \iff [\Delta, \mathcal{Q}^{\lambda,\lambda}(K)] = A\Delta$ .

**Remark 3.3.3.** We can obtain explicit expressions for the higher symmetries of  $\Delta^\ell$  via the formulae for the conformally equivariant quantization given in [38], and in (2.1) for  $\ell = 1$ . Eastwood proposed in [18] a generalization of the formulae giving higher symmetries of the Laplacian in the curved case, but quoting him "it is difficult to say whether they are symmetry operators of the conformal Laplacian". Resorting to results of Silhan [38], our method lead also

to a formula for higher symmetries of any conformal powers of the Laplacian in the curved case. This is a work in progress to determine whether they are indeed symmetry operators.

#### 4. ALGEBRAS OF SYMMETRIES: GEOMETRIC REALIZATIONS AND DEFORMATIONS

The aim of this section is to provide geometric interpretations to the algebras of classical and quantum symmetries, as well as identifying the star product induced by the composition of quantum symmetries. In all this section we work over a conformally flat manifold  $(M, [g])$ .

**4.1. Algebras of symmetries are generated by  $\mathfrak{g}$ .** Let us give a brief reminder on universal enveloping algebra  $\mathfrak{U}(\mathfrak{g})$  and symmetric algebra  $S(\mathfrak{g})$  of an arbitrary Lie algebra  $\mathfrak{g}$ . From the tensor algebra of  $\mathfrak{g}$ , they inherit respectively a filtration  $\{\mathfrak{U}_k(\mathfrak{g})\}_k$  and a graduation  $S(\mathfrak{g}) = \bigoplus_k S_k(\mathfrak{g})$  such that  $\text{gr } \mathfrak{U}(\mathfrak{g}) \simeq S(\mathfrak{g})$ . Consequently, the canonical projections  $\mathfrak{U}_k(\mathfrak{g}) \rightarrow \mathfrak{U}_k(\mathfrak{g})/\mathfrak{U}_{k-1}(\mathfrak{g})$  define a principal symbol map, whose right inverses are called quantization of  $S(\mathfrak{g})$ . The symmetrization map  $\text{Sym} : S(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{g})$  defined by

$$\text{Sym} : X_{i_1} \cdots X_{i_k} \mapsto \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} X_{\tau(i_1)} \cdots X_{\tau(i_k)}$$

is known to define a  $\mathfrak{g}$ -equivariant quantization of  $S(\mathfrak{g})$  for the canonical extensions of the adjoint action of  $\mathfrak{g}$  to  $S(\mathfrak{g})$  and  $\mathfrak{U}(\mathfrak{g})$ . Any other  $\mathfrak{g}$ -equivariant quantization is then of the form  $\Phi = \text{Sym} \circ \phi$ , with  $\phi = \text{Id} + N$  and  $N$  a  $\mathfrak{g}$ -equivariant map on  $S(\mathfrak{g})$  lowering the degree. Analogously to the case of symbols, a  $\mathfrak{g}$ -equivariant graded star product  $\star_\Phi$  can be obtained on  $S(\mathfrak{g})$ , as the pull-back of the product on  $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$  by the map  $\Phi_\hbar = (\Phi \otimes \text{Id}) \circ \mathfrak{S}$ , where  $\mathfrak{S} : S(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$  is the linear map defined by  $(i\hbar)^k \text{Id}$  on  $S_k(\mathfrak{g})$ . Denoting by  $\tau$  and  $\gamma$  the anti-automorphisms of  $\mathfrak{U}(\mathfrak{g})$  and  $S(\mathfrak{g})$  defined by  $-\text{Id}$  on  $\mathfrak{g}$ , the symmetry of the star product on  $S(\mathfrak{g})$  is equivalent to  $\Phi_\hbar(\bar{\cdot}) = \tau \circ \Phi_\hbar(\cdot)$ , or simply  $\Phi \circ \gamma = \tau \circ \Phi$ .

We return to the case  $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$ . Via the defining universal properties of the both algebras  $S(\mathfrak{g})$  and  $\mathfrak{U}(\mathfrak{g})$ , the pull-back defined by the moment map  $\mu^* : \mathfrak{g} \rightarrow \mathcal{S}_1^0$  and the Lie derivative  $\ell^\lambda : \mathfrak{g} \rightarrow \mathcal{D}_1^{\lambda, \lambda}$  extend to algebra morphisms  $\mu^* : S(\mathfrak{g}) \rightarrow \mathcal{S}^0$  and  $\ell^\lambda : \mathfrak{U}(\mathfrak{g}) \rightarrow \mathcal{D}^{\lambda, \lambda}$ .

**Theorem 4.1.1.** *Let  $\lambda \in \mathbb{R}$ . The space  $\mathcal{K}$  of classical symmetries is equal to the algebra  $\mu^*(S(\mathfrak{g})) \simeq S(\mathfrak{g})/I$ , with  $I$  a graded ideal of  $S(\mathfrak{g})$ . Its image by the conformally equivariant quantization, namely  $\mathcal{A}^\lambda := \mathcal{Q}^{\lambda, \lambda}(\mathcal{K})$ , is an algebra satisfying  $\mathcal{A}^\lambda = \ell^\lambda(\mathfrak{U}(\mathfrak{g}))/J^\lambda$ , with  $J^\lambda$  a filtered ideal such that  $\text{gr } J^\lambda \simeq I$ . The star product  $\star_\lambda$  induced by  $\mathcal{Q}^{\lambda, \lambda}$  on  $\mathcal{S}^0$  restricts then to a  $\mathfrak{g}$ -equivariant graded star product on  $\mathcal{K}$ .*

Moreover, the conformally equivariant quantization of  $\mathcal{K}$  lifts to a  $\mathfrak{g}$ -equivariant quantization  $\Phi^\lambda$  of  $S(\mathfrak{g})$ , such that the following diagram commutes

$$(4.1) \quad \begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\Phi^\lambda} & \mathfrak{U}(\mathfrak{g}) \\ \mu^* \downarrow & & \downarrow \ell^\lambda \\ \mathcal{K} & \xrightarrow{\mathcal{Q}^{\lambda, \lambda}} & \mathcal{A}^\lambda \end{array}$$

*Proof.* We start with proving  $\mathcal{K} = \mu^*(S(\mathfrak{g}))$ . Since they are  $\mathfrak{g}$ -modules, this amounts to prove  $\mu^*(S(\mathfrak{g})) \cap \mathcal{S}_{k,s}^0 = \mathcal{K}_{k,s}$  for all  $k, s$ . Let  $\ell \in \mathbb{N}^*$ . The space  $\mathcal{A}^{\lambda,\ell}$  is a subalgebra of  $\mathcal{D}^{\lambda,\lambda}/(\Delta^\ell)$  and the principal symbol map commutes with the product, so we get that  $\mathcal{K}^\ell$  is a subalgebra of  $\text{Pol}(T^*\mathbb{R}^n)/(R^\ell)$ . Hence,  $\mathcal{K}$  is a subalgebra of  $\text{Pol}(T^*\mathbb{R}^n)$ , which contains the algebra  $\mu^*(S(\mathfrak{g}))$  generated by the conformal Killing vector fields, in particular  $\mu^*(S(\mathfrak{g})) \cap \mathcal{S}_{k,s}^0 \subset \mathcal{K}_{k,s}$  for all  $k, s$ . Using the elements generated by constant Killing vector fields, we get that  $\mu^*(S(\mathfrak{g})) \cap \mathcal{S}_{k,s}^0$  is non-empty and since  $\mathcal{K}_{k,s}$  is irreducible, we are done. As  $\mu^*$  respects the grading, its kernel  $I$  is a graded ideal.

Now, we prove that  $\mathcal{A}^\lambda = \ell^\lambda(\mathfrak{U}(\mathfrak{g}))$ . By semi-simplicity of  $\mathfrak{g}$ , its finite dimensional representations are reducible. In particular, for any  $k$ ,  $\mathcal{K} \cap \mathcal{S}_k^0$  can be viewed as a submodule of  $S_k(\mathfrak{g})$ , leading to the decomposition  $S(\mathfrak{g}) \simeq \mathcal{K} \oplus I$  of the symmetric algebra. In other words,  $\mu^*$  admits a  $\mathfrak{g}$ -equivariant section. Using the embedding of  $\ell^\lambda(\mathfrak{U}(\mathfrak{g}))$  into  $\mathcal{D}^{\lambda,\lambda}$ , we get then the following diagram of  $\mathfrak{g}$ -modules

$$\begin{array}{ccc}
 S(\mathfrak{g}) & \xrightarrow{\text{Sym}} & \mathfrak{U}(\mathfrak{g}) \\
 \uparrow \mathcal{K} & & \downarrow \ell^\lambda \\
 & & \ell^\lambda(\mathfrak{U}(\mathfrak{g})) \\
 & \searrow \mathcal{Q}^{\lambda,\lambda} & \downarrow \\
 & & \mathcal{D}^{\lambda,\lambda}
 \end{array}$$

Each arrow in the latter diagram is  $\mathfrak{g}$ -equivariant and preserves the principal symbol. Hence, uniqueness of  $\mathcal{Q}^{\lambda,\lambda}$  implies that it is commutative, proving  $\mathcal{A}^\lambda = \ell^\lambda(\mathfrak{U}(\mathfrak{g}))$ . Since  $\ell^\lambda$  preserves the filtration, its kernel  $J^\lambda$  is filtered. Using the commutativity of the following diagram,

$$\begin{array}{ccc}
 \mathfrak{U}_k(\mathfrak{g}) & \longrightarrow & \mathcal{A}^\lambda \cap \mathcal{D}_k^{\lambda,\lambda} \\
 \downarrow & & \downarrow \\
 S_k(\mathfrak{g}) & \longrightarrow & \mathcal{K}_k
 \end{array}$$

where the vertical arrows denote principal symbol maps, we get that  $\text{gr } J^\lambda = I$ .

We have proved  $S(\mathfrak{g}) \simeq \mathcal{K} \oplus I$ , and along the same line we get  $\mathfrak{U}(\mathfrak{g}) \simeq \mathcal{A}^\lambda \oplus J^\lambda$ . Using again the semi-simplicity of  $\mathfrak{g}$ , the isomorphism  $J_k^\lambda/J_{k-1}^\lambda \simeq I_k$  leads to  $J_k^\lambda \simeq I_k \oplus J_{k-1}^\lambda$ . Thus, it exists an isomorphism of  $\mathfrak{g}$ -modules between  $I$  and  $J^\lambda$ , inverse to the symbol map. Together with the previous decomposition of  $S(\mathfrak{g})$  and  $\mathfrak{U}(\mathfrak{g})$ , this ensures the existence of the quantization  $\Phi^\lambda$  and the commutativity of the diagram (4.1).  $\square$

The proof shows that  $\mathcal{K}_{k,s} = \mu^*(S(\mathfrak{g})) \cap \mathcal{S}_{k,s}^0$ . Thus, the  $s$ -generalized conformal Killing  $k$ -tensors are algebraically generated from the conformal Killing vectors, i.e. they are all reducible over a conformally flat manifold. This widely generalizes a result in [37], stating the reducibility of second order conformal Killing tensors on conformally flat manifolds.

**Corollary 4.1.2.** *Let  $\ell \in \mathbb{N}^*$  and  $\lambda = \frac{n-2\ell}{2n}$ . The following diagram of  $\mathfrak{g}$ -modules is commutative*

$$(4.2) \quad \begin{array}{ccc} \mathcal{K} & \xrightarrow{\mathcal{Q}^{\lambda,\lambda}} & \mathcal{A}^\lambda \\ \downarrow & & \downarrow \\ \mathcal{K}^\ell = \mathcal{K}/(R^\ell) & \xrightarrow{\mathcal{Q}^{\lambda,\lambda}} & \mathcal{A}^\lambda/(\Delta^\ell) = \mathcal{A}^{\lambda,\ell} \end{array}$$

The algebra of  $s$ -generalized conformal Killing tensors, for  $s < \ell$ , satisfies  $\mathcal{K}^\ell \simeq \mathcal{S}(\mathfrak{g})/I^\ell$ , with  $I^\ell$  the ideal generated by  $I$  and  $(\mu^*)^{-1}(R^\ell)$ , and the algebra of higher symmetries of  $\Delta^\ell$  satisfies  $\mathcal{A}^{\lambda,\ell} \simeq \mathfrak{U}(\mathfrak{g})/J^{\lambda,\ell}$  with  $J^{\lambda,\ell}$  the ideal generated by  $J^\lambda$  and  $(\ell^\lambda)^{-1}(\Delta^\ell)$ . The star product  $\star_\lambda$  projects accordingly onto  $\mathcal{K}^\ell$ , defining there a graded  $\mathfrak{g}$ -equivariant star product.

*Proof.* By definition  $\mathcal{K}^\ell = \mathcal{K}/(R^\ell)$  and by Theorem 3.3.1 we have  $\mathcal{Q}^{\lambda,\lambda}(\mathcal{K}^\ell) = \mathcal{A}^{\lambda,\ell}$ . Adding that  $R^\ell \in \mu^*(\mathcal{S}(\mathfrak{g}))$  and  $\mathcal{Q}^{\lambda,\lambda}((R^\ell)) = (\Delta^\ell)$ , we get the announced commutative diagram. The remaining results trivially follow.  $\square$

**4.2. A family of coadjoint orbits of  $O(p+1, q+1)$ .** We restrict in this paragraph to the case where  $M$  is the homogeneous space  $\mathbb{S}^p \times \mathbb{S}^q$  of the conformal group  $G = O(p+1, q+1)$ . The latter acts linearly and in a Hamiltonian way on  $T^*\mathbb{R}^{p+1,q+1}$ , and forms a Howe dual pair with  $SL(2, \mathbb{R})$ , its centralizer in the symplectic linear group  $Sp(2n+2, \mathbb{R})$ . More precisely, they form a symplectic dual pair in the sense of [3]. Denoting their moment maps by

$$\mu : T^*\mathbb{R}^{p+1,q+1} \longrightarrow \mathfrak{g}^* \quad \text{and} \quad J : T^*\mathbb{R}^{p+1,q+1} \longrightarrow \mathfrak{sl}(2, \mathbb{R})^*,$$

the symplectic reduction of  $T^*\mathbb{R}^{p+1,q+1}$  w.r.t.  $SL(2, \mathbb{R})$  for different regular values of the moment map  $J$  gives rise to (finite coverings of) all the coadjoint orbits in the image of  $\mu$ . We rather describe them as symplectic reductions at 0 w.r.t. Lie subgroups of  $SL(2, \mathbb{R})$  generated by the flow of Hamiltonian functions in  $J^*(\mathcal{S}(\mathfrak{sl}(2, \mathbb{R})))$ , i.e. polynomial functions in  $x^2 = \eta_{AB}x^Ax^B$ ,  $xp = x^Ap_A$  and  $p^2 = \eta^{AB}p_Ap_B$ , where  $(x^A, p_A)$  are cartesian coordinates on  $T^*\mathbb{R}^{p+1,q+1}$ . For example, the Casimir elements of  $\mathfrak{g}$  and  $\mathfrak{sl}(2, \mathbb{R})$  in  $\mathcal{C}^\infty(T^*\mathbb{R}^{p+1,q+1})$  are equal to  $C = (xp)^2 - x^2p^2$  and  $C/4$  respectively, if we define the Killing form by the map  $(X, Y) \mapsto \frac{1}{2}\text{Tr}(\rho(X)\rho(Y))$  with  $\rho$  their standard representation.

We denote by  $\langle f_1, \dots, f_k \rangle$  the Lie group generated by the flow of Hamiltonian functions  $f_1, \dots, f_k \in \mathcal{C}^\infty(T^*\mathbb{R}^{p+1,q+1})$  and by  $T^*\mathbb{R}^{p+1,q+1} // \langle f_1, \dots, f_k \rangle$  the corresponding symplectic quotient at 0. If those functions are linearly closed under the Poisson bracket, the latter space is then the quotient of the zero locus of  $f_1, \dots, f_k$  by their Hamiltonian flows. By the Marsden-Weinstein theorem, this quotient space is a symplectic manifold if 0 is a regular value of the involved Hamiltonian functions. E.g., we have

$$(4.3) \quad T^*(\mathbb{R}^{p+1,q+1} \setminus \{0\}) // \langle xp, x^2 \rangle \simeq T^*M.$$

Notice that  $T^*M$  splits in three stable submanifolds under the Hamiltonian  $G$ -action, according to the sign of the norm of the covectors, with straightforward notations:  $T^*M = T_+^*M \sqcup T_0^*M \sqcup T_-^*M$ .

**Theorem 4.2.1.** *Let  $p, q \geq 1$ ,  $n \geq 3$  and  $P(\alpha, \beta)$  be the space of planes in  $\mathbb{R}^{p+1, q+1}$  of signature  $(\alpha, \beta)$ . The coadjoint orbits of  $G$  in the image of  $\mu$  are classified along four families:*

- (1) *the one parameter family of semi-simple orbits  $\mathcal{O}_{a+}$  and  $\mathcal{O}_{a-}$  for  $a \in \mathbb{R}_+^*$  such that*

$$T^*\mathbb{R}^{p+1, q+1} // \langle xp, C - a \rangle \xrightarrow{\mathbb{Z}_2} \mathcal{O}_{a+} \sqcup \mathcal{O}_{a-} \xrightarrow{\simeq} P(2, 0) \sqcup P(0, 2),$$

- (2) *the one parameter family of semi-simple orbits  $\mathcal{O}_a$  for  $a \in \mathbb{R}_-^*$  such that*

$$T^*\mathbb{R}^{p+1, q+1} // \langle xp, C - a \rangle \xrightarrow{\mathbb{Z}_2} \mathcal{O}_a \xrightarrow{\simeq} P(1, 1),$$

- (3) *the two nilpotent orbits  $\mathcal{O}_{0+}$  and  $\mathcal{O}_{0-}$  such that*

$$T_+^*M \sqcup T_-^*M \xrightarrow{\mathbb{Z}_2} \mathcal{O}_{0+} \sqcup \mathcal{O}_{0-} \xrightarrow{\mathbb{R}^*} P(1, 0) \sqcup P(0, 1),$$

- (4) *The minimal nilpotent orbit  $\mathcal{O}_{00}$  such that*

$$(T^*M \setminus M) // \langle R \rangle \xrightarrow{\mathbb{Z}_2} \mathcal{O}_{00} \xrightarrow{\mathbb{R}^*} P(0, 0).$$

*All the arrows denote  $G$ -equivariant coverings, whose fibers are indicated as superscript. The first ones are symplectomorphisms.*

*Proof.* Through the  $G$ -modules isomorphisms  $\Lambda^2\mathbb{R}^{p+1, q+1} \simeq \mathfrak{g} \simeq \mathfrak{g}^*$ , coadjoint orbits are identified to  $G$ -orbits in the space of bivectors, endowed with the natural  $G$ -action. The moment map  $\mu$  is then given by  $T^*\mathbb{R}^{p+1, q+1} \ni (u, v) \mapsto u \wedge v$ , and valued in the space of simple bivectors  $\text{Bv} = \{u \wedge v \mid u, v \in \mathbb{R}^{p+1, q+1}\}$ . Our key tool is the  $G$ -equivariant projection of  $\text{Bv}$  on the Grassmannian  $\text{Gr}(2, n+2)$  of planes in  $\mathbb{R}^{p+1, q+1}$ . This is encompassed in the following sequence of  $G$ -spaces:

$$(4.4) \quad T^*\mathbb{R}^{p+1, q+1} \xrightarrow{\text{SL}(2, \mathbb{R})} \text{Bv} \xrightarrow{\mathbb{R}^*} \text{Gr}(2, n+2) \cup \{0\},$$

where the superscripts denote the fibers of the coverings over  $\text{Gr}(2, n+2)$ . The moment map preserves the Poisson structure, hence a  $G$ -stable submanifold of  $T^*\mathbb{R}^{p+1, q+1}$  projects onto coadjoint orbits of  $G$ , which themselves project onto  $G$ -orbits of  $\text{Gr}(2, n+2) \cup \{0\}$ . Thanks to the Witt Theorem, the latter are known to be  $\{0\}$  and the 6 spaces  $P(\alpha, \beta)$  of planes of given signature  $(\alpha, \beta)$  for the induced metric. E.g., the  $G$ -stable embedding of  $T_+^*M \sqcup T_-^*M$  into  $T^*\mathbb{R}^{p+1, q+1}$  projects onto coadjoint orbits, which project onto the orbits  $P(1, 0) \sqcup P(0, 1)$  in  $\text{Gr}(2, n+2)$ . The fibers of these two projections are easily computed. In the three other cases, the zero locus of the given Hamiltonian functions have the announced images in  $\text{Gr}(2, n+2)$ . Their Hamiltonian flows act only in the fibers of  $\mu$ , so the map  $\mu$  descends to the symplectic quotients. The fibers are proved to reduce to  $\mathbb{Z}_2$ , resorting to the explicit expressions of the one parameter groups generated by  $xp$ ,  $C$  and  $R$ . Namely, they respectively are  $(u, v) \mapsto (tu, t^{-1}v)$ ,  $(u, v) \mapsto (u + (tv^2)v, v - (tu^2)u)$  and  $(u, v) \mapsto (u + tv, v)$  for  $t \in \mathbb{R}^*$ . We end with the four sequences (1), (2), (3), and (4). There, a unique coadjoint orbit lies over each orbit in  $\text{Gr}(2, n+2)$ , since the action of the group  $G$  is transitive in the fibers of each arrow. For a proof of the minimality of  $\mathcal{O}_{00}$  we refer to [41].  $\square$

The two last points in the latter theorem combine, according to Cordani [12], to provide a conformal regularization by  $T^*M$  of the cone  $\mathcal{O}_{0+} \cup \mathcal{O}_{00} \cup \mathcal{O}_{0-}$ , with singularity in  $\mathcal{O}_{00}$ .

**Remark 4.2.2.** *The used symplectic reductions of  $T^*\mathbb{R}^{p+1,q+1}$  correspond clearly to symplectic reduction w.r.t.  $\mathrm{SL}(2, \mathbb{R})$  at, respectively, the points  $(0, \sqrt{a}, \pm\sqrt{a})$ ,  $(0, -\sqrt{|a|}, -\sqrt{|a|})$ ,  $(0, 0, \pm a)$  and  $(0, 0, 0)$ . Hence, we obtain a bijection between the coadjoints orbits of  $\mathrm{SL}(2, \mathbb{R})$  and the ones in the image of  $\mu$ , in accordance with [3].*

Now, we determine the algebra of regular functions on each coadjoint orbit of  $G$  in the image of  $\mu$ . Using ambient description, we have  $\mathfrak{g} \simeq \Lambda^2 \mathbb{R}^{p+1,q+1}$ , that we represent by the following Young diagram  $\square$ . Accordingly, elementary representation theory of the orthogonal Lie algebra leads to

$$(4.5) \quad \mathfrak{g} \odot \mathfrak{g} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}_0 \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}_0 \oplus \mathbb{R}.$$

In the second decomposition, the index 0 denotes the trace free part, and the three components correspond to  $\mathcal{K}_{2,0}$ ,  $\mathcal{K}_{2,1}$  and the one-dimensional space generated by the Casimir element in  $S_2(\mathfrak{g})$ , still denoted by  $C$ . The extra term in the decomposition of  $\mathfrak{g} \odot \mathfrak{g}$  is generated by exterior products in  $\Lambda \mathbb{R}^{p+1,q+1}$  of elements of  $\mathfrak{g}$ .

**Lemma 4.2.3.** *The kernel of the pull-back  $\mu^* : S(\mathfrak{g}) \rightarrow \mathcal{C}^\infty(T^*\mathbb{R}^{p+1,q+1})$  by the moment map of  $\mathfrak{g}$  is the ideal generated by  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$ .*

*Proof.* Since elements of  $\mathfrak{g}$  are skew-symmetric 2-tensors  $V^{AB}$  on  $\mathbb{R}^{p+1,q+1}$ , the map  $\mu^*$  is explicitly given by  $V^{AB \cdots CD} = x_A \cdots x_C V^{AB \cdots CD} p_B \cdots p_D$ , and vanishes then on tensors  $V^{AB \cdots CD}$  which are skew-symmetric in any 3 indices. Hence,  $\mu^*(S_k(\mathfrak{g}))$  is a submodule of the one given by the Young diagram with 2 lines and  $k$  columns. They turn to be equal. Indeed, the irreducible components of the latter Young diagram correspond via  $\mu^*$  to product of pure trace and traceless tensors, clearly none of them is in the kernel.

Let  $T^{ABCDEF} \in \square \odot \square$ . By definition, its skew-symmetrization over 3 indices pertaining to  $ABCD$  or  $CDEF$  vanishes. Then, straightforward computations prove that so is it over any 3 indices in  $ABCDEF$ , i.e.  $\square \odot \square = \square$ . The same holds for Young diagrams of greater length, and we finally get that the algebra  $\mu^*(S(\mathfrak{g}))$ , or the one of two lines Young diagrams, is isomorphic to  $S(\mathfrak{g}) / \left( \begin{array}{|c|} \hline \square \\ \hline \end{array} \right)$ .  $\square$

**Proposition 4.2.4.** *Let  $a \in \mathbb{R}$  and  $\mathcal{I}_a = \left( [C - a]\mathbb{R} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \right)$ . The algebras of regular functions are given by  $S(\mathfrak{g})/\mathcal{I}_a$  on  $\mathcal{O}_{a(\pm)}$ , and  $S(\mathfrak{g})/\mathcal{I}_{00}$  on  $\mathcal{O}_{00}$ , with  $\mathcal{I}_{00} = (\square_0) + \mathcal{I}_0$ . Moreover, the algebras of regular functions on  $\mathcal{O}_{0\pm}$  and  $\mathcal{O}_{00}$  are isomorphic to  $\mathcal{K}$  and  $\mathcal{K}^1$  respectively.*

*Proof.* According to (4.3), we get that  $T_\pm^*(\mathbb{R}^{p+1,q+1} \setminus \{0\}) // \langle xp, C \rangle$  is a  $\mathbb{Z}_2$ -covering of  $T_\pm^*M$ . Thus, each coadjoint orbit of  $G$  in the image of  $\mu$  is finitely covered by a symplectic reduction of  $T^*\mathbb{R}^{p+1,q+1}$  and its algebra of regular functions is isomorphic to the corresponding reduction of  $\mu^*(S(\mathfrak{g}))$ . The reduction w.r.t.  $xp$  modifies only the fibers of  $\mu$  and the Casimir  $C$  Poisson

commutes with all elements in  $\mu^*(S(\mathfrak{g}))$ , so that reduction w.r.t.  $\langle xp, C - a \rangle$  amounts to modding out by  $(C - a)$ . This gives the result for the orbits  $\mathcal{O}_{a(\pm)}$  for every  $a \in \mathbb{R}$ . By Theorem 4.1.1, the algebra  $\mathcal{K}$  is generated by  $\mathfrak{g}$ . The isomorphism  $\mathcal{K} \simeq S(\mathfrak{g})/\mathcal{I}_0$  follows then from the local diffeomorphism between  $T_{\pm}^*M$  and  $\mathcal{O}_{0\pm}$ .

Resorting to Theorem 4.2.1, the coadjoint orbit  $\mathcal{O}_{00}$  is locally diffeomorphic to the symplectic quotient  $(T^*M \setminus M) // \langle R \rangle$ , i.e. to the quotient of the submanifold  $T_0^*M \setminus M$  of null covectors by the Hamiltonian flow of  $R$ . Thus, the algebra of regular functions on  $\mathcal{O}_{00}$  arises as a reduction of  $\mathcal{K}$ . Since  $\{R, \mathcal{K}\} \subset (R)$ , this reduced algebra is  $\mathcal{K}/(R)$ , which is isomorphic to  $\mathcal{K}_1$ . As  $R \in \mathcal{K}_{2,1}$  is the pull-back of an element in  $\square_{0,1}$ , we finally obtain  $\mathcal{I}_{00}$ .  $\square$

**Remark 4.2.5.** *The kernel  $\mathcal{I}_0$  of  $\mu^* : S(\mathfrak{g}) \rightarrow C^\infty(T^*M)$  is generated as an ideal and  $\mathfrak{g}$ -module by the two relations  $X_{ij}X_{kl} - X_{ik}X_{jl} + X_{il}X_{jk} = 0$  and  $\eta^{ij}X_{ik}X_j^k + 2X_0^2 - 2\eta^{ij}X_i\bar{X}_j = 0$  (i.e.  $\mu^*C = 0$ ) on  $T^*M$ , where we use the following notation for the generators of  $\mathfrak{g}$ :  $X_i = p_i$ ,  $X_{ij} = x_i p_j - x_j p_i$ ,  $X_0 = x^i p_i$  and  $\bar{X}_i = x^j x_j p_i - 2x^i x^j p_j$ .*

**4.3. The Joseph ideal.** We return now to a general conformally flat manifold  $(M, [g])$ , and use notation of Paragraph 4.1, in particular  $\mathcal{K} \simeq S(\mathfrak{g})/I$  and  $\mathcal{K}^1 \simeq S(\mathfrak{g})/I^1$ . The algebraic description of these symmetry algebras obtained in Proposition 4.2.4 are of local nature and thus carry on over  $(M, [g])$ . Hence, the ideals  $I, I^1$  identify to  $\mathcal{I}_0, \mathcal{I}_{00}$ . We compute now the corresponding ideals  $J^\lambda$  and  $J^{\lambda,1}$  in  $\mathfrak{U}(\mathfrak{g})$ , which define the algebras  $\mathcal{A}^\lambda$  and  $\mathcal{A}^{\lambda,1}$ . Recall that we define the Killing form by  $\frac{1}{2}\text{Tr}(XY)$ , for every  $X, Y \in \mathfrak{g}$ . The corresponding Casimir operator  $\mathcal{C}$  in  $\mathfrak{U}(\mathfrak{g})$  is given by the symmetrization of the Casimir element  $C$  in  $S(\mathfrak{g})$ .

**Proposition 4.3.1.** *For every  $\lambda \in \mathbb{R}$ , the ideals  $J^\lambda$  are equal to  $(\text{Sym}(\square) \oplus [C - \rho(\lambda)]\mathbb{R})$ , where  $\mathcal{C}$  is the Casimir operator of  $\mathfrak{g}$  and  $\rho(\lambda) = n^2\lambda(1 - \lambda)$  its eigenvalue on  $\lambda$ -densities.*

*For  $\lambda = \frac{n-2}{n}$ ,  $J^{\lambda,1} = (\text{Sym}(\square_{0,1})) + J^\lambda$  is the Joseph ideal.*

*Proof.* Since the graded ideal associated to  $J^\lambda$  is  $I$ , we deduce that  $J^\lambda$  is also quadratic and resorting to Theorem 4.1.1, we have  $J_2^\lambda = \Phi^\lambda(I_2)$  with  $\Phi^\lambda = \text{Sym} \circ \phi^\lambda$ . The map  $\phi^\lambda$  being  $\mathfrak{g}$ -equivariant, the space  $\phi^\lambda(I_2)$  is a  $\mathfrak{g}$ -submodule of  $\mathbb{R} \oplus \mathfrak{g} \oplus S_2(\mathfrak{g})$ . Hence,  $J_2^\lambda$  is generated by  $\text{Sym}(\square)$  and the Casimir operator  $\mathcal{C}$  of  $\mathfrak{U}(\mathfrak{g})$ , modified by some real number. Since this element projects onto 0 on  $\mathcal{D}^{\lambda,\lambda}$ , this real number is necessarily the eigenvalue of  $\ell^\lambda(\mathcal{C})$  on  $\lambda$ -densities. The latter has been computed in [15], where the opposite Killing form is used. The formula giving  $J^{\lambda,1}$  is trivially deduced from Corollary 4.1.2. Thanks to Theorem 3.3.1, we have the isomorphism of algebras  $\text{gr } \mathfrak{U}(\mathfrak{g})/J^{\lambda,1} \simeq S(\mathfrak{g})/I^1$ . This implies that  $J^{\lambda,1}$  is completely prime and its characteristic variety is the closure of the minimal nilpotent coadjoint orbit of  $G$ . These properties characterize the Joseph ideal [24].  $\square$

The identification of the Joseph ideal in the context of the higher symmetries of the Laplacian was already noticed in [17, 39], but via the concordance of explicit expressions, rather than from its characteristic properties. From the latter proposition, the ideals  $J^{\lambda,\ell}$  associated to symmetries of  $\Delta^\ell$  are straightforwardly deduced as they are generated by  $J^\lambda$



and  $(\ell^\lambda)^{-1}(\Delta^\ell)$ . The latter corresponds to the Young diagram  $\boxed{\dots\Box}_0$  of length  $2\ell$ . The determination of those ideals has been already performed in the context of higher symmetries of  $\Delta^\ell$  in [18, 19, 23], but in different terms. Let us make clear the link between the two approaches. We denote by  $\langle \cdot, \cdot \rangle$  the chosen Killing form and  $C$  the associated Casimir element in  $S(\mathfrak{g})$ . In the previous works, the projections of  $X \odot Y \in \mathfrak{g} \odot \mathfrak{g}$  on each irreducible component are used. Following  $\mathfrak{g} \odot \mathfrak{g} = \boxplus_0 \oplus \boxplus_0 \oplus \mathbb{R} \oplus \boxplus$ , we have  $X \odot Y = X \boxtimes Y + X \bullet Y + \frac{\langle X, Y \rangle}{2 \dim \mathfrak{g}} C + X \wedge Y$ . Then, the ideal  $J^\lambda$  is clearly generated by  $\text{Sym}(\frac{\langle X, Y \rangle}{2 \dim \mathfrak{g}}(C - \rho(\lambda)) + X \wedge Y)$  for  $X, Y \in \mathfrak{g}$  or equivalently by

$$\text{Sym}\left(X \odot Y - X \boxtimes Y - X \bullet Y + \frac{\rho(\lambda)}{2 \dim \mathfrak{g}} \langle X, Y \rangle\right),$$

which is the obtained expression in [18, 19, 23], modulo the extra generator associated to  $R^\ell$ .

**4.4. Quantization of a family of coadjoint orbits of  $G$ .** We have described the algebras of regular functions on the coadjoint orbits of  $G$  in the image of  $\mu$  as quotients  $S(\mathfrak{g})/\mathcal{I}$  for various ideals  $\mathcal{I}$ . We perform now their deformation quantization, which should arise from quantization-like maps  $S(\mathfrak{g})/\mathcal{I} \rightarrow \mathcal{U}(\mathfrak{g})/\mathcal{J}$ , for ideals  $\mathcal{J}$  to be defined. This means to find deformations of the Poisson algebra  $S(\mathfrak{g})$  which restrict to coadjoint orbits, and this is in general not trivial. Thus, for  $\mathfrak{g}$  a semi-simple Lie algebra, Cahen, Gutt and Rawnsley have proved the non-existence of a star product on  $\mathcal{C}^\infty(\mathfrak{g}^*)$  tangent to each coadjoint orbit [9]. Here, for the graded algebras  $\mathcal{K}$  and  $\mathcal{K}^1$ , such a quantization map is provided by the conformally equivariant quantization, we have proved that it induces a graded  $\mathfrak{g}$ -equivariant star-product on them and determines the corresponding ideals  $\mathcal{J}$ . As the remaining algebras  $S(\mathfrak{g})/\mathcal{I}_a$  for  $a \neq 0$  are only filtered, this direct approach fails and we resort to the following Lemma.

**Lemma 4.4.1.** *Let  $a \in \mathbb{R}$ . There exists a  $\mathfrak{g}$ -equivariant linear map  $\phi_a = \text{Id} + N_a$  on  $S(\mathfrak{g})$ , such that  $N_a$  lowers the degree and  $\phi_a(\mathcal{I}_a) = \mathcal{I}_0$ , hence  $S(\mathfrak{g})/\mathcal{I}_a \simeq \mathcal{K}$ .*

*Proof.* We know that  $S(\mathfrak{g}) \simeq I \oplus \mathcal{K}$  and  $I = (C) + \left(\boxplus\right)$ . Resorting to the semi-simplicity of  $\mathfrak{g}$  and the filtration of  $\mathcal{I}_a = (C - a) + \left(\boxplus\right)$ , we get that  $S(\mathfrak{g}) \simeq \mathcal{I}_a + S(\mathfrak{g})/\mathcal{I}_a$  and  $(C - a)$  admits a  $\mathfrak{g}$ -stable complement in  $\mathcal{I}_a$ . The map  $\phi_a$  defined by  $\frac{C}{C-a} \text{Id}$  on  $(C - a)$  and the identity on a  $\mathfrak{g}$ -stable complementary space satisfies the required properties.  $\square$

**Theorem 4.4.2.** *There exists a family of  $\mathfrak{g}$ -equivariant quantizations  $(\Phi_a^\lambda)_{a, \lambda \in \mathbb{R}}$  of  $S(\mathfrak{g})$  such that: (i) it lifts  $(\mathcal{Q}^{\lambda, \lambda})_{\lambda \in \mathbb{R}}$  to  $S(\mathfrak{g})$  for  $a = 0$ , (ii) it induces a family of symmetric  $\mathfrak{g}$ -invariant star products on the coadjoint orbits  $\mathcal{O}_{a(\pm)}$  for  $a \in \mathbb{R}$ , (iii) if  $a = 0$  and  $\lambda = \frac{n-2}{2n}$ , it induces the unique graded  $\mathfrak{g}$ -equivariant star-product on  $\mathcal{O}_{00}$ , introduced by Arnal-Benhamor-Cahen [1] and Astashkevich-Brylinski [2].*

*Proof.* The Theorem 4.1.1 ensures the existence of a  $\mathfrak{g}$ -equivariant quantization  $\Phi_\lambda$  of  $S(\mathfrak{g})$  lifting  $\mathcal{Q}^{\lambda, \lambda}$  for every  $\lambda \in \mathbb{R}$ . The lift property is equivalent to  $\Phi^\lambda(I) = J^\lambda$ . We define then the family of  $\mathfrak{g}$ -equivariant quantizations  $\Phi_a^\lambda = \Phi^\lambda \circ \phi_a$ , where  $\phi_a$  is introduced in Lemma 4.4.1. It can be chosen such that  $\phi_0 = \text{Id}$ , so (i) is trivially satisfied. The Lemma 4.4.1 ensures

that  $\Phi_a^\lambda(\mathcal{I}_a)$  is an ideal and a  $\mathfrak{g}$ -module, hence the  $\mathfrak{g}$ -invariant star product  $\star_{\Phi_a^\lambda}$  on  $S(\mathfrak{g})$ , induced by  $\Phi_a^\lambda$ , descends on the quotient  $S(\mathfrak{g})/\mathcal{I}_a$ . We recall that  $\star_{\Phi_a^\lambda}$  is symmetric if  $\Phi_a^\lambda$  satisfies  $\tau \circ \Phi_a^\lambda = \Phi_a^\lambda \circ \gamma$ . Redefining  $\Phi_a^\lambda$  by  $\frac{1}{2}(\Phi_a^\lambda + \tau \circ \Phi_a^\lambda \circ \gamma)$  this is trivially the case, and the quantization  $\Phi_0^\lambda$  is still a lift of  $\mathcal{Q}^{\lambda,\lambda}$  by uniqueness of the latter. This proves (ii). The last point follows then from Corollary 4.1.2, Proposition 4.2.4 and the uniqueness result in [1, 2].  $\square$

**Remark 4.4.3.** *For two distinct coadjoint orbits, the star products obtained above do not coincide in general. This is reminiscent to the work of Fioresi and Lledo [21], dealing with star products tangential to semi-simple coadjoint orbits of semi-simple Lie groups.*

Quantization of a coadjoint orbit of a Lie group is more usually understood as building an irreducible unitary representation of this group from that orbit. Since the Hamiltonian  $G$ -action on  $T^*(\mathbb{S}^p \times \mathbb{S}^q)$  preserves its vertical polarization, the Kirillov's orbit method [25] applies to the coadjoint orbits  $\mathcal{O}_{0^\pm}$ . It leads to the unitary representation of  $G$  onto the pre-Hilbert space of compactly supported half-densities on  $\mathbb{S}^p \times \mathbb{S}^q$ , endowed with its canonical Hermitian product  $(\phi, \psi) = \int_M \bar{\phi} \psi$ . The corresponding infinitesimal action of the Lie algebra  $\mathfrak{g}$  by Lie derivative extends to a deformed representation of the whole Poisson algebra  $\mathcal{K}$  of regular functions on  $\mathcal{O}_{0^\pm}$ , via the conformally equivariant quantization  $\mathcal{Q}^{\frac{1}{2}, \frac{1}{2}}$ .

In contradistinction, the minimal coadjoint orbit cannot be quantize via the orbit method as it admits no  $G$ -invariant polarization. This key fact is proved in [40] and stems from the following result of [6]:  $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$  can be identified to a Lie subalgebra of polynomial vector fields on  $\mathbb{R}^m$  iff  $m \geq p+q$ . The following proposition shows how conformally equivariant quantization provides a supplement to the orbit method for the minimal nilpotent orbit.

**Proposition 4.4.4.** *Let  $\lambda = \frac{n-2}{2n}$ . The algebra  $\mathcal{K}^1$  of conformal Killing tensors on  $\mathbb{S}^p \times \mathbb{S}^q$  identifies to the one of regular functions on the minimal coadjoint orbit  $\mathcal{O}_{00}$ . Its conformally equivariant quantization  $\mathcal{Q}^{\lambda,\lambda}(\mathcal{K}^1) \simeq \mathfrak{U}(\mathfrak{g})/J^{\lambda,1}$  is the algebra of higher symmetries of the conformal Laplacian on  $\mathbb{S}^p \times \mathbb{S}^q$ , which acts faithfully on its kernel. Moreover,  $J^{\lambda,1}$  is the Joseph ideal.*

*Proof.* Since the Joseph ideal is a maximal primitive ideal, the action of  $\mathcal{Q}^{\lambda,\lambda}(\mathcal{K}^1) \simeq \mathfrak{A}^{\lambda,1}$  on the kernel of  $\Delta$  is trivial or faithful, hence it is faithful.  $\square$

In particular, the infinitesimal action of  $\mathfrak{g}$  on harmonic functions is given by the Lie derivative of  $\frac{n-2}{2n}$ -densities and the kernel of the induced representation of  $\mathfrak{U}(\mathfrak{g})$  is the Joseph ideal. This result dates back to [5] and was so far the only link between the minimal coadjoint orbit of  $G$  and its minimal representation, which integrates the one of  $\mathfrak{g}$  on  $\ker \Delta$ .

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